The homology group of the square 1 puzzle

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Several papers have been devoted to the group theoretical description of Rubik's cube-like puzzles [BH], [GT], [L], and [Si]. These give rise to excellent examples and projects in undergraduate group theory courses. The square 1 puzzle is different in that the moves do not preserve the original cube shape of the puzzle. In this sense, square 1 is more closely related to the famous 15 puzzle (which changes "shape" since the blank square moves around) than the Rubik's cube.

This paper studies the group theoretic properties of the collection G of all "words" in the basic moves of the square 1 puzzle which preserve the cube shape. This collection G forms a group which, motivated by [W], we call the homology group of the square 1 puzzle. The list of shapes which the square 1 puzzle can make is given in [Sn]. It is not hard to see that the homology group of any one of these other shapes is conjugate to G, so from a group-theoretic standpoint, we may focus our attention on the cube.

1. The main result

Let S_n denote the symmetric group of degree n, i.e., the group of permutations of $\{1, 2, ..., n\}$. Let $sgn : S_n \to \{\pm 1\}$ denote the homomorphism which assigns to each permutation its sign (the sign of a cyclic permutation of length r is $(-1)^{r+1}$, for example).

We shall see that the size of the homology group of the square 1 is about .8 billion.

Theorem 1.1. *G is isomorphic to the kernel of index 2 in $S_8 \times S_8$ of the homomorphism $f : S_8 \times S_8 \to \{\pm 1\}$ defined by $f(g_1, g_2) = sgn(g_1)sgn(g_2)$. Consequently,*

$|G| = 2^{13}3^4 5^2 7^2 = 812851200$.

As a corollary of the proof of this theorem, given below, we shall see that any even permutation of the corners is possible and any even permutation of the wedges is possible.

Let H denote the enlarged square 1 group generated by all legal moves preserving the cube shape and all illegal moves (i.e., disassembly and reassembly is allowed) preserving the cube shape. It is clear that

$$H \cong S_8 \times S_8.$$

1.1. Some notation

We shall assume that the puzzle is in the solved position with the "square 1" side in front, right-side up. Let

- u denote rotation of the up face by 30^o clockwise,
- d denote rotation of the up face by 30^o clockwise,
- R denote rotation of the cube by 180^o though one of the skew-diagonal cuts (in a given position, at most one such move is possible, so this is unambiguous).

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Like the 15 puzzle, and unlike the Rubik's cube, not any sequence of u, d, and R's is possible.

Let

$$T(x,y) = u * R * x * y * R * u^{-1}, B(x,y) = d^{-1} * R * x * y * R * d,$$

where x, y are moves of the square 1 puzzle.

In the notation of these diagrams, we have

$$\begin{gathered}
uRu^{-1}d^{-1}Rd = (2,8)(4,6) \\
T(u^3,1) = (1',6',7',4')(1,6,7,4) \\
T(1,d^3) = (2',3',8',5')(2,3,8,5) \\
B(u^3,1) = (1,2,7,8)(1',6',7',4') \\
B(1,d^3) = (3,4,5,6)(2',3',8',5').
\end{gathered}$$

1.2. Two subgroups

Let

$$G_u =< T(u^3,1), T(1,d^3) >$$

and

$$G_d =< B(u^3,1), B(1,d^3) >$$

Lemma 1.2. *G_u and G_d are each isomorphic to $C_4 \times C_4$.*

proof: We have $T(u^3,1)T(1,d^3) = T(1,d^3)T(u^3,1)$. Moreover, $T(u^3,1)$ and $T(1,d^3)$ are each of order 4. Since

$$C_4 \times C_4 =< a,b \mid a^4 = 1, b^4 = 1, ab = ba >,$$

the lemma follows. $\square$

The homology group of the square 1 puzzle is defined to be

$$G =< d^3, u^3, B(u^3,1), B(1,d^3), T(u^3,1), T(1,d^3) >$$

We shall use the following labelings to describe the moves of the square 1 puzzle

2. Proof of the theorem

We shall prove the theorem in the following steps:

- Show that the wedge 3-cycle $(1,2,3)$ and the corner 3-cycle $(1',2',3')$ each belong to G.
- Show that any wedge 3-cycle $(1,2,i)$ and each corner 3-cycle $(1',2',i')$ belong to G.
- Show that there is a injective homomorphism $\phi : G \to S_8 \times S_8$ where the image $\phi(G)$ contains $A_8 \times A_8$.
- Conclude that $G \cong S_8 \times S_8/\{\pm 1\}$.

Step 1: First, we claim that $(1,2,3)$ belongs to G. In fact, the 3-cycle $(1,2,3)$ is obtained from the move

$$M_1 = (B(u^3,1)*d^3)*((B(u^3,1)*d^{-3})*(B(u^{-3},1)*T(1,d^{-3})*d^6)))^4*(B(u^3,1)*d^3)^{-1}.$$

(Incidently, this 80 move long manuever may be verified using GAP [Gap]. See also [Sn].)

Next, we claim that $(1',2',3')$ belongs to G. In fact, $M_2 = Ru^3Rd^{-3}Ru^3(Ru^{-3})^2d^3Ru^{-3}$ is the product of 2-cycles $(2',3')(3,4)$. (This move was found in [Sn].) Therefore, $u^3M_2u^{-3}$ is the product of 2-cycles $(1',2')(2,3)$. The product of these is $(1',2',3')(2,3,4)$. Since $(2,3,4)$ is obtained from $u^{-3}M_1u^3$, we see that $(1',2',3')$ is in G. (This may also be verified using GAP.)

Step 2: Let g be any move in G which sends wedges 3 to wedge i, resp., and does not move wedges $1,2$ (it may permute other wedges and corners). Then $(1,2,i) = g*(1,2,3)*g^{-1}$. Thus $(1,2,i) \in G$.

The proof that each $(1',2',i') \in G$ is similar.

Step 3: It is clear from our definition that there is an injection $G \to S_8 \times S_8$ *as sets*. The verification that this is a homomorphism is straightforward.

Step 4: The group A_8 is generated by the 3-cycles $(1, 2, i)$ (see [W], for example). Since these all belong to G, all even wedge permutations are possible. Similarly, all even corner permutations are possible. Thus $A_8 \times A_8 \subset G$.

Let $p_1 : S_8 \times S_8 \to S_8$ denote the projection onto the first factor. Let p_2 denote the projection onto the second factor. For each generator $g \in \{d^3, u^3, B(u^3, 1), B(1, d^3), T(u^3, 1), T(1, d^3)\}$ of G we have $sgn(p_1(g)) = sgn(p_2(g))$. Thus the image $\phi(G)$ is strictly contained in $S_8 \times S_8$. In fact, this shows that $\phi(G)$ is contained in the kernel $ker(f)$ of the homomorphism $f : S_8 \times S_8 \to \{\pm 1\}$ defined in the statement of the theorem. Since

$$A_8 \times A_8 \subset G \subset ker(f),$$

$[ker(f) : A_8 \times A_8] = 2$, and $T(u^3, 1) \notin A_8 \times A_8$, the theorem follows. □

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